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EQUATIONS OF KINETICS FOR AGGREGATION PROCESSES IN SUSPENSIONS

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A. S. POPEL, S. A. REGIRER and N. Kh. SHADRINA
(Moscow, Leningrad)
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Generalized kinetic equations for determining the aggregation of particles in suspensions are derived with allowance for dispersion and multiple and exchange interactions. A system of equations is derived in a general form for moments of the distribution function, and a method for determining equilibrium distribution is indicated. Some exact solutions, including self-similar, of the proposed kinetic equations are obtained.

Physical properties of many suspensions substantially depend on the processes of aggregation and dispersion of suspended particles. Such processes are defined by special kinetic equations, an example of which is the equation of drop coagulation (see [1]). The latter takes into account only one aggregation process, viz., the amalgamation of drops produced by double collisions. Theories which take into account also the dispersion of particles (see, e.g., [2]) are known. However for some systems with high concentration of suspended particles such as, for instance, blood in which erythrocytes occupy about half of the volume, it is necessary to take into consideration a more complex interaction between particles.

Thus in a concentrated suspension the determining effect may be that of collisions other than double, which in the case of blood become significant for an erythrocyte concentration $H \geqslant 5\%$ [3]. Besides aggregation and dispersion of particles, exchange interactions are possible when two or more particles not identical to the original ones are formed as the result of collisions (*). If under certain conditions there exists a limit dimension for the aggregate but with possible collisions of arbitrary particles, exchange interactions must necessarily occur.

The above phenomena are taken into consideration in the kinetic equation which is derived and analyzed below in Sects. 1 – 4 and 7. Certain exact solutions of that equation are presented in Sects. 5 and 6.

1. The kinetic equation. Let us consider a suspension in the form of a mixture of a "carrier" fluid and suspended particles which may coalesce into aggregates of any arbitrary form as the result of effective collisions, i.e. leading to the sticking to-

*) This was brought to the attention of the authors by A. G. Kulikovskii.

gether of particles. Aggregates may be dispersed by the action of external, mainly hydrodynamic forces or because of instability. The possibility of exchange interaction is assumed.

We define the state of the mixture by the distribution function $f(v, t, \mathbf{r})$ ($f(v)$ in abbreviated notation) such that $f(v)\,dv$ is the expected number of aggregates of volumes comprised between v and $v + dv$ in a unit of the physical space volume. The integrals

$$n = \int_0^\infty f(v)\,dv, \qquad H = \int_0^\infty vf(v)\,dv \tag{1.1}$$

represent the numerical and volume concentrations of aggregates, respectively. The average volume of an aggregate is defined by

$$w = H / n \tag{1.2}$$

The fundamental equation satisfied by the distribution function $f(v)$ is of the form

$$\frac{df(v)}{dt} \equiv \frac{\partial f}{\partial t} + \mathbf{u}\frac{\partial f}{\partial \mathbf{r}} = \Gamma_k^+ - \Gamma_k^- + \Gamma_f^+ - \Gamma_f^- + \Gamma_e^+ - \Gamma_e^- - \operatorname{div}\mathbf{q} \tag{1.3}$$

where $\mathbf{u}$ is the mean velocity of the mixture, $\Gamma = \Gamma(v, t, \mathbf{r})$ is the rate of distribution change produced by the aggregation processes, subscripts k, f and e relate, respectively, to sticking, dispersion and exchange, and superscripts $\pm$ denote the formation and disappearance of v-aggregates (of volume close to v), respectively. The stream $\mathbf{q}$ represents diffusion of aggregates, and the integrals

$$\mathbf{q}_n = \int_0^\infty \mathbf{q}\,dv, \qquad \mathbf{q}_H = \int_0^\infty v\mathbf{q}\,dv \tag{1.4}$$

have the meaning of flow density of numerical and volume concentrations, respectively. Denoting the average velocity of v-aggregates by $\mathbf{U}(v)$, we have

$$\mathbf{q} = (\mathbf{u} - \mathbf{U}(v))\,f \tag{1.5}$$

We denote by $K_s(m_1, \ldots, m_s) \equiv K_s(m_s|_1^s)$ the probability of formation of an aggregate of volume $M_s \equiv m_1 + \ldots + m_s$ produced by a simultaneous sticking of s aggregates of volumes $m_1, \ldots, m_s$. By definition function K_s takes into account the probability of the s-multiple collision itself. The symbol $m\,|_1^s$ will henceforth denote the set of arguments $m_1, \ldots, m_s$.

We denote by $F_s(m\,|_1^s)$ the probability density of dispersion of an aggregate of volume M_s into s parts of volumes $m_1, \ldots, m_s$, i.e. $F_s dm_1 \ldots dm_s$ defines the probability of simultaneous formation of fragments of volumes contained in the intervals $(m_1, m_1 + dm_1), \ldots, (m_s, m_s + dm_s)$.

Finally, the probability density of exchange interaction consisting of instantaneous transformation of the set of aggregates of volumes $m_1, m_2, \ldots, m_s$ into a set of volumes $p_1, p_2, \ldots, p_r$ is denoted by $E_{sr}(m\,|_1^s;\ p\,|_1^r)$, where $M_s = P_r \equiv p_1 + \ldots + p_r$, and $s > 1$.

Functions K_s and F_s are symmetric and for negative arguments are identically zero; function E_{sr} has similar properties with respect to variables m_i and p_i taken individually. We assume that K_s, F_s and E_{sr} may also depend explicitly on time t and coordinates $\mathbf{r}$.

Using the derived probability functions and taking into account their properties, we can write for $\Gamma_k^{\pm}$, $\Gamma_f^{\pm}$ and $\Gamma_e^{\pm}$ the following expressions:

$$\Gamma_k^+ = \sum_{s=2}^{\infty} \frac{1}{s!}\int\ldots\int K_s(m\,|_1^{s-1}, v - M_{s-1}) f(v - M_{s-1}) \prod_{j=1}^{s-1} f(m_j)\, dm_j \qquad (1.6)$$

$$\Gamma_k^- = \sum_{s=2}^{\infty} \frac{1}{(s-1)!}\int\ldots\int K_s(m\,|_1^{s-1}, v) f(v) \prod_{j=1}^{s-1} f(m_j)\, dm_j$$

$$\Gamma_f^+ = \sum_{s=2}^{\infty} \frac{1}{(s-1)!}\int\ldots\int F_s(m\,|_1^{s-1}, v) f(M_{s-1} + v) \prod_{j=1}^{s-1} dm_j \qquad (1.7)$$

$$\Gamma_f^- = \sum_{s=2}^{\infty} \frac{1}{s!}\int\ldots\int F_s(m\,|_1^{s-1}, v - M_{s-1}) f(v) \prod_{j=1}^{s-1} dm_j$$

$$\Gamma_e^+ = \sum_{s=2}^{\infty}\sum_{r=2}^{\infty} \frac{1}{s!\,(r-1)!}\int\ldots\int E_{sr}(m\,|_1^{s-1}, P_{r-1} + v - M_{s-1};\; p\,|_1^{r-1}, v) \;\times \qquad (1.8)$$

$$f(P_{r-1} + v - M_{s-1}) \prod_{j=1}^{s-1} f(m_j)\, dm_j \prod_{k=1}^{r-1} dp_k$$

$$\Gamma_e^- = \sum_{s=2}^{\infty}\sum_{r=2}^{\infty} \frac{1}{(s-1)!\,r!}\int\ldots\int E_{sr}(m\,|_1^{s-1}, v;\; p\,|_1^{r-1}, M_{s-1} +$$

$$v - P_{r-1}) f(v) \prod_{j=1}^{s-1} f(m_j)\, dm_j \prod_{k=2}^{r-1} dp_k$$

Unless otherwise stated, here and subsequently integration is to be carried out with respect to all variables from 0 to ∞. Equation (1.3) together with formulas (1.6)–(1.8) and corresponding boundary conditions determine the distribution function for specified K_s, F_s, E_{sr} and $\mathbf{q}$. In the particular case, when

$$K_s \equiv 0 \quad (s \geqslant 3), \quad F_s \equiv 0 \quad (s \geqslant 2), \quad E_{sr} \equiv 0 \quad (s \geqslant 2,\; r \geqslant 2), \qquad (1.9)$$
$$\mathbf{q} = 0$$

we obtain from (1.3), (1.6) – (1.8) the equation derived in [4].

2. Moments of the distribution function. We call

$$Q_q = \frac{1}{n}\int v^q f(v)\, dv \qquad (2.1)$$

a moment of order q.

Multiplying Eq. (2.3) by v^q and integrating with respect to v from 0 to ∞, after transformation, we obtain

$$\frac{d}{dt} nQ_q = \sum_{s=2}^{\infty}\frac{1}{s!}\int\ldots\int K_s(m\,|_1^s)\Big[\Big(\sum_1^s m_i\Big)^q - \qquad (2.2)$$

$$\sum_1^s m_i^q\Big]\prod_{j=1}^{s} f(m_j)\, dm_j - \sum_{s=2}^{\infty}\frac{1}{s!}\int\ldots\int F_s(m\,|_1^s)\Big[\Big(\sum_1^s m_i\Big)^q -$$

$$\sum_1^s m_i^q\Big]f(M_s)\prod_{j=1}^s dm_j + \sum_{s=2}^\infty \sum_{r=2}^\infty \frac{1}{s!r!}\int\ldots\int E_{sr}(m|_1^s; p|_1^r) \times$$

$$\Big[\sum_2^r p_i^q - \sum_1^s m_i^q\Big]\prod_{j=1}^s f(m_j)\,dm_j \prod_{k=1}^{r-1} dp_k - \operatorname{div}\int v^q \mathbf{q}(v)\,dv$$

$$(p_r = M_s - P_{r-1})$$

The double sum appearing here may also be represented in the form

$$\sum_{s=1}^\infty \frac{1}{s!}\int\ldots\int\Big[E_s^+(m|_1^s)\prod_{j=1}^s f(m_j) - E_s^-(m|_1^s) f(M_s)\Big]\times$$

$$\Big[\Big(\sum_1^s m_i\Big)^q - \sum_1^s m_i^q\Big]\prod_{j=1}^s dm_j$$

$$E_s^+ = \sum_{r=2}^\infty \frac{1}{r!}\int\ldots\int E_{sr}(m|_1^s; p|_1^r)\prod_{k=1}^{r-1} dp_k \tag{2.3}$$

$$E_s^- = \sum_{r=2}^\infty \frac{1}{r!}\int\ldots\int E_{sr}(p|_1^r; m|_1^s) f(p_r) f^{-1}(P_r)\prod_{k=1}^{r-1} f(p_k)\,dp_k$$

It follows from (2.2) that for $q = 0$ the equation which defines the variation of the complete number of aggregates is

$$\frac{dn}{dt} = \sum_{s=2}^\infty \frac{1-s}{s!}\int\ldots\int\Big[K_s(m|_1^s)\prod_{j=1}^s f(m_j) - \tag{2.4}$$

$$F_s(m|_1^s) f(M_s)\Big]\prod_{j=1}^s dm_j + \sum_{s=2}^\infty\sum_{r=2}^\infty \frac{r-s}{s!\,r!}\times$$

$$\int\ldots\int E_{sr}(m|_1^s; p|_1^r)\prod_{j=1}^s f(m_j)\,dm_j \prod_{k=1}^{r-1} dp_k - \operatorname{div}\mathbf{q}_n$$

All exchange terms with $r = s$ in (2.4), obviously, vanish, since such interactions do not alter the number of aggregates. For $q = 1$ all integral terms in the moment equation (2.2) vanish, and we obtain the conventional diffusion equation

$$dH/dt = -\operatorname{div}\mathbf{q}_H \tag{2.5}$$

In the particular case, when conditions (1.9) are satisfied, we obtain from (2.2) the equation derived in [5].

To explain the essence of above transformations let us consider, for instance, the integral

$$\frac{1}{s!\,(r-1)!}\int\ldots\int v^q E_{sr}(m|_1^{s-1}, P_{r-1}+v-M_{s-1};\ p|_1^{r-1}, v)\times \tag{2.6}$$

$$f(P_{r-1}+v-M_{s-1})\prod_{j=1}^{s-1} f(m_j)\,dm_j \prod_{k=1}^{r-1} dp_k dv =$$

$$\frac{1}{s!\,(r-1)!}\int\ldots\int p_r^q E_{sr}(m|_1^s;\ p|_1^r)\prod_{i=1}^s f(m_j)\,dm_j \prod_{k=1}^{r-1} dp_k$$

$$(p_r = v = M_s - P_{r-1})$$

By substituting the variable of integration p_r for p_l, where $1 \leqslant l \leqslant r-1$, we obtain

$$\frac{1}{s!\,(r-1)!}\int\cdots\int p_r{}^q E_{sr}(m\,|_1^s;\ p\,|_1^r)\prod_{j=1}^{s} f(m_j)\,dm_j\prod_{k=1}^{l-1}dp_k\prod_{k=l+1}^{r}dp_k \tag{2.7}$$

From this, after converting $p_r \to p_l$ and $p_l \to p_r$, we obtain

$$\frac{1}{s!\,(r-1)!}\int\cdots\int p_l{}^q E_{sr}(m\,|_1^s;\ p\,|_1^r)\prod_{j=1}^{s} f(m_j)\,dm_j\prod_{k=1}^{r-1}dp_k \tag{2.8}$$

Thus the considered integral can be presented in the form (2.8), where $l = 1, 2, \ldots, r$ (see (2.7)). Adding all of the different r representations, we obtain

$$\frac{1}{s!\,r!}\int\cdots\left(\sum_1^r p_i{}^q\right)E_{sr}(m\,|_1^s;\ p\,|_1^r)\prod_{j=1}^{s} f(m_j)\,dm_j\prod_{k=1}^{r-1}dp_k$$

Formula (2.3) is derived as follows. Let us express the double sum in (2.2) in the form

$$\sum_{s=2}^{\infty}\sum_{r=2}^{\infty}\frac{1}{s!\,r!}E_{sr}(m\,|_1^s;\ p\,|_1^r)\left[\left(\sum_1^s m_i\right)^q-\sum_1^s m_i{}^q\right]\prod_{j=1}^{s} f(m_j)\times$$
$$dm_j\prod_{k=1}^{r-1}dp_k-\sum_{s=2}^{\infty}\sum_{r=2}^{\infty}\frac{1}{s!\,r!}\int\cdots\int E_{sr}(m\,|_1^s;\ p\,|_1^r)\left(\left[\sum_1^r p_i\right)^q-\right.$$
$$\left.\sum_1^r p_i{}^q\right]\prod_{j=1}^{s} f(m_j)\,dm_j\prod_{k=1}^{r-1}dp_k$$

We carry out the conversion in the second sum according to the rule $m_j \to p_k$, $d_k \to m_j$, $s \to r$ and $r \to s$, and transpose the order of summation and integration. This yields

$$-\sum_{s=2}^{\infty}\sum_{r=2}^{\infty}\frac{1}{s!\,r!}\int\cdots\int E_{rs}(p\,|_1^r;\ m\,|_1^s)\left[\left(\sum_1^s m_i\right)^q-\sum_1^s m_i{}^q\right]\prod_{k=1}^{r} f(p_k)\,dp_k\sum_{j=1}^{s-1}dm_j$$

Substituting m_s for the variable of integration p_r and taking into account the equality $M_s = P_r$, we obtain

$$-\sum_{s=2}^{\infty}\sum_{r=2}^{\infty}\frac{1}{s!\,r!}\int\cdots\int E_{rs}(p\,|_1^r;\ m\,|_1^s)\left[\left(\sum_1^s m_i\right)^q-\sum_1^s m_i{}^q\right]\times$$
$$f(P_r)\sum_{k=1}^{r-1} f(p_k)\,dp_k\prod_{j=1}^{s}dm_j$$

Further conversion to (2.3) is obvious.

3. Aggregation equilibrium. In the case of absence of diffusion Eq. (2.2) with allowance for (2.3) can be written as

$$\frac{d}{dt}nQ_q=\sum_{s=2}^{\infty}\frac{1}{s!}\int\cdots\int\left[K_s{}^*(m\,|_1^s)\prod_{j=1}^{s} f(m_j)-F_s{}^*(m\,|_1^s)\,f(M_s)\right]\times \tag{3.1}$$
$$\left[\left(\sum_1^s m_i\right)^q-\sum_1^s m_i{}^q\right]\prod_{j=1}^{s}dm_j$$
$$K_s{}^*=K_s+E_s{}^+,\qquad F_s{}^*=F_s+E_s$$

Let us consider a system with an interaction of only one multiplicity l

$$K_s \equiv 0 \quad (s \neq l), \quad F_s \equiv 0 \quad (s \neq l), \quad E_{sr} \equiv 0 \quad (s \neq l, \quad r \neq l) \tag{3.2}$$

For such system from (3.1) we obtain

$$\frac{d}{dt} nQ_q = \frac{1}{l!} \int \cdots \int \Big[K_l^* (m|_1^l) \prod_{j=1}^{l} f(m_j) - F_l^*(m|_1^l) f(M_l) \times$$
$$\Big[\Big(\sum_1^l m_i\Big)^q - \sum_1^l m_i^q \Big] \prod_{j=1}^{l} dm_j \tag{3.3}$$

The following statement – which can be checked by direct computation – is valid: if the functional equation (*)

$$K_l^* (m|_1^l) \prod_{j=1}^{l} f(m_j) - F_l^* (m|_1^l) f(M_l) = 0 \tag{3.4}$$

has solutions such that $df / dt = 0$, these solutions are also solutions of the input kinetic equation (1.3) for the particular case of (3.2).

It is thus possible to determine equilibrium distributions and analyze conditions of their existence without having to solve the input kinetic equation of the form (1.3) (**)

It should be noted, however, that the question of whether the solutions of the fundamental equation (3.4) determine all of the aggregate equilibrium states possible with the considered system, and whether the vanishing of all derivatives dnQ_q / dt is equivalent to equilibrium in the meaning of df / dt remains open.

If interactions of different multiplicities occur simultaneously in the system, equilibrium states are also possible, but attempts at finding a similar example proved unsuccessful.

4. Discrete systems. Let us assume that only particles whose volumes belong to the sequence $v|_1 = v_1, v_2, \ldots, v_k, \ldots, (v_k > v_{k-1})$ exist in the suspension, and that $v_i + v_j$ also belong to $v|_1$ for any i and j. The distribution function is then of the form

$$f(v) = \sum_{i=1}^{\infty} n_i \delta (v - v_i) \tag{4.1}$$

where n_i is the numerical concentration of aggregates of volume v_i, and $H_i = n_i v_i$ is their volume concentration. Then

$$n = \sum_1^{\infty} n_i, \quad H = \sum_1^{\infty} n_i v_i, \quad nQ_q = \sum_1^{\infty} n_i (v_i)^q \tag{4.2}$$

For a discrete distribution to exist it is necessary that

$$F_s (m|_1^s) = 0, \quad E_{sr} (m|_1^s; p|_1^r) = 0$$

even if only one of the quantities m_i or p_i, respectively, do not belong to the sequence $v|_1$. Functions F_s and E_{sr} behave with respect to m_i and p_i, respectively, as δ-functions

*) The theory of such equations and effective methods of their solution are given in [6].

**) In the case of more specifically defined systems Eq. (3.4) is amenable to thermodynamic interpretation (see [7]).

of order $s-1$ and $r-1$, e.g.,

$$F_s(m|_1^s) = \frac{1}{s}\Psi_s(m|_1^s)\sum_{(k)}\sum_{j=1}^{s}\delta^{-1}(m_j - v_{l_j}^{(k)})\prod_{i=1}^{s}\delta(m_i - v_{l_i}^{(k)}) \quad (4.3)$$

where Ψ_s is a regular symmetric function which vanishes for negative arguments, and $v_{l_1}^{(k)}, v_{l_2}^{(k)}, \ldots, v_{l_s}^{(k)}$ is the kth arbitrarily chosen set of s terms from $v|_1$. Sets k and k' are considered to be different when $v_{l_i}^{(k)} \neq v_{l_i}^{(k')}$ even for a single value of i. External summation in (4.3) is extended to all different sets.

It should be noted that in computing Γ (see Sect. 1) it is generally necessary to bear in mind as regards the quantity of formed or vanished v aggregates that the result of sticking or dispersion depends on the number of particles of volume v taking part in such events. Furthermore, physically identical elements of volume space are repeatedly taken into account in the integration of formulas for Γ. The factorial coefficients at integrals adjust the computation exactly only at points where all arguments of integrated functions are different. Consequently, functions

$$K_s^*(m|_1^s) = K_s(m|_1^s)\,\Theta(m|_1^s), \quad F_s^\times(m|_1^s) = F_s(m|_1^s)\,\Theta(m|_1^s) \quad (4.4)$$
$$E_{sr}^\times(m|_1^s;\ p|_1^r) = E_{sr}(m|_1^s;\ p|_1^r)\,\Theta(m|_1^s)\,\Theta(|p|_1^r)\,\Theta(m|_1^s) = s!\,\psi^{-1}(m|_1^s)$$

where $\psi(m|_1^s)$ is the number of physically different permutations in the sequence $m|_1^s$, should have been used in formulas (1.6) – (1.8) instead of K_s, F_s and E_{sr}. It can be shown with the use of elementary combinatorial considerations that the introduction of these functions takes completely into account both of the above aspects.

Since in the N-dimensional space function $\Theta(m|_1^N)$ differs from unity in a manifold of not more than $N-1$ dimensions, hence for any arbitrary bounded functions

$$\int\cdots\int[\Theta(m|_1^N)-1]\;A(m|_1^N)\prod_{j=1}^{N}dm_j = 0$$

Owing to this property, the absence of the correction factor Θ does in no way affect the input kinetic equation, as long as the distribution functions in the integrands and the interaction properties are bounded. However, function Θ must be taken into account when passing to the definition of discrete systems with distributions of the kind (4.1). It is then necessary to set formally

$$\int\cdots\int[\Theta(m|_1^N)-1]\,\delta(m|_1^N - m'|_1^N)\prod_{j=1}^{N}dm_j = \Theta(m'|_1^N)-1$$

where the right-hand part is evidently not always zero.

These considerations are not new, although they have not been explicitly formulated in the literature, and in many papers are either not used at all or applied, for example, to dispersion terms of the kinetic equation (see, e.g., [8]).

Thus, for passing to the equations of the discrete system it is necessary to substitute $K_s^\times$, $F_s^\times$ and $E_{sr}^\times$ in (1.6) – (1.8) for K_s, F_s and E_{sr} and then, carry out integration of Eq. (1.3)

$$\int_{v_i-\alpha}^{v_i+\alpha}\frac{df}{dt}\,dv = \int_{v_i-\alpha}^{v_i+\alpha}[\Gamma_k^+ - \Gamma_k^- + \Gamma_f^+ - \Gamma_f^- + \Gamma_e^+ - \Gamma_e^- - \operatorname{div}\mathbf{q}]\,dv \quad (4.5)$$

where α is defined by the inequalities

$$v_{i-1} < v_i - \alpha < v_i < v_i + \alpha < v_{i+1}$$

The general form of obtained equations is very cumbersome, hence we restrict these to the example of a system with double interactions (1.9) and aggregates formed from identical "elementary" particles. Setting

$$v_i = iw_0, \quad \Theta(v_i, v_j) = 1 + \delta_{ij}$$

$$F_2(m_1, m_2) = \frac{1}{2}\Psi(m_1, m_2)\sum_{i,j}[\delta(m_1 - v_i) + \delta(m_2 - v_j)]$$

$$E_{22}(m_1, m_2; p_1, p_2) = \frac{1}{2}\varepsilon(m_1, m_2; p_1, p_2)\sum_{i,j}[\delta(p_1 - v_i) + \delta(p_2 - v_j)]$$

we obtain

$$\frac{dn_j}{dt} = \frac{1}{2}\sum_{k=1}^{j-1} n_k n_{j-k}\alpha_{k,j-k} - n_j\sum_{k=1}^{\infty} n_k\alpha_{k,j} + \sum_{k=j+1}^{\infty} n_k\beta_{k-j,j} - \frac{1}{2}n_j\beta_j + \frac{1}{2}\sum_{l=1}^{\infty}\sum_{k=1}^{\infty} n_k n_l \gamma_{l,k,l+k-j,j} - \frac{1}{2}n_j\sum_{l=1}^{\infty} n_l\gamma_{l,j} - \operatorname{div}\mathbf{q}_j \tag{4.6}$$

$$\alpha_{k,l} = K_2(v_k, v_l)(1 + \delta_{kl}), \quad \beta_{kl} = \Psi(v_k, v_l)(1 + \delta_{kl})$$

$$\beta_j = \sum_{i=1}^{j-1}\beta_{i,j-1}$$

$$\gamma_{k,l,i,j} = \varepsilon(v_k, v_l; v_i, v_j)(1 + \delta_{kl})(1 + \delta_{ij}), \quad \gamma_{l,j} = \sum_{i=1}^{l+j-1}\gamma_{l,j,l+j-i,i}$$

For $\mathbf{q}_j = 0$, $\beta_{k,l} = 0$ and $\gamma_{k,l,i,j} = 0$ this yields the classic Smoluchowski equations [9], while for nonzero $\mathbf{q}_j$, $\alpha_{k,l}$, and $\beta_{k,l}$, and $\gamma_{k,l,i,j} = 0$ the equations proposed in [2, 8] are obtained.

5. Exact solutions. Let us now consider certain possibilities of obtaining exact solutions of the kinetic equation (1.3) or moment equations (2.2). We restrict the analysis to the spatially homogeneous problem of aggregation in the absence of dispersion, exchange interactions, and diffusion, i.e., to the problem analyzed in considerable detail for the case of double collisions. In the case of collisions of l-multiplicity from (1.3) and (1.4), we obtain

$$\frac{df}{dt} = \frac{1}{l!}\int\ldots\int\left[K_l(m|_1^{l-1}, v - M_{l-1})f(v - M_{l-1}) - lK_l(m|_1^{l-1}, v)f(v)\right]\prod_{j=1}^{l-1} f(m_j)\,dm_j \tag{5.1}$$

$$\frac{dn}{dt} = \frac{1-l}{l!}\int\ldots\int K_l(m|_1^{l})\prod_{j=1}^{l} f(m_j)\,dm_j \tag{5.2}$$

We would point out that for integral $\alpha_1, \alpha_2, \ldots$ the following equalities

$$\int\ldots\int m_1^{\alpha_1} m_2^{\alpha_2}\ldots m_{s-1}^{\alpha_{s-1}}\eta(v - M_{s-1})(v - M_{s-1})^{\alpha_s}e^{-pv}f(v -$$

$$M_{s-1})\prod_{j=1}^{s-1} f(m_j)\,dm_j dv = \prod_{j=1}^{s} (-1)^{\alpha_j} \frac{\partial^{\alpha_j}\varphi}{\partial p^{\alpha_j}}$$

$$\int\dots\int m_1^{\alpha_1} m_2^{\alpha_2}\dots m_{s-1}^{\alpha_{s-1}} v^{\alpha_s} e^{-pv} \prod_{j=1}^{s-1} f(m_j)\,dm_j f(v)\,dv =$$

$$\left[\prod_{j=1}^{s-1}(-1)^{\alpha_s}\frac{\partial^{\alpha_j}\varphi}{\partial p^{\alpha_j}}\right](-1)^{\alpha_j}\frac{\partial^{\alpha_s}\varphi}{\partial p^{\alpha_s}}, \quad \varphi(p) = \int_0^\infty e^{-pv} f(v)\,dv$$

$$\eta(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geqslant 0 \end{cases}$$

are valid.

Hence for any polynomial functions K_l (which vanish for negative arguments) by applying to the input equation (1.3) (for $q = 0$, $F_s = 0$, $E_{sr} = 0$) the Laplace transformation with respect to the variable v, we obtain an equation in φ, which is of the first order with respect to time, while its order with respect to the variable p is equal to the highest power of the variable m_i in K_l $(m|_1^l)$. The coefficients of the equation contain the a priori unknown derivatives of φ with respect to p for $p = 0$, i.e. the moments of distribution function are of an order not exceeding that of the equation. Setting in the equation $p \to 0$, we obtain an additional relationship linking moments of various orders, which is, generally, insufficient for the determination of all coefficients. Obviously the equation in φ and the additional relationship form a closed system only when the equation is of the first order with respect to p. In that case we obtain a system containing only n, φ and H, where H is determined by the input data. This implies that the described procedure can be effective, if K_l is a linear function with respect to each of m_i separately, i.e. it is a sum of elementary symmetric polynomials

$$K_l = K_{l0} + K_{l1}\sum_1^l m_i + K_{l2}\sum_{\substack{i,j=1\\ j\neq i}}^{l} m_i m_j + \dots + K_{ll}\prod_1^l m_i \tag{5.3}$$

where for K_{lk} the sum C_l^k of various products appears as a factor.

It is evident that integrals of the form

$$\int\dots\int m_1^{\alpha_1} m_2^{\alpha_2}\dots m_s^{\alpha_s}\left[\left(\sum_1^s m_i\right)^q - \sum_1^s m_i^q\right]\prod_{j=1}^{s} f(m_j)\,dm_j$$

can be expressed in terms of distribution function moments of an order not exceeding $q - 1 + \max\alpha_i$, when $q > 0$, and not higher than $\max\alpha_i$ when $q = 0$. Hence the substitution of (5.3) $(\max\alpha_i = 1)$ into the moment equation (2.2) yields a system of unconnected equations which have to be solved consecutively. All these considerations can be readily extended to the system in which collisions of various multiplicities take place.

In particular the equation for numerical concentration of aggregates assumes the form

$$\frac{dn}{dt} = \sum_{s=1}^{\infty}\frac{1-s}{s!}\sum_{i=0}^{s} C_s^i K_{si} n^{s-i} H^i \tag{5.4}$$

Examples. In the case of collisions of l-multiplicity and $K_l = K_{l_0} = \mathrm{const}$

from (5.4) we obtain
$$\frac{dn}{dt} = -\frac{l-1}{l!}K_{l_0}n^l \tag{5.5}$$

Applying the Laplace transformation to Eq. (1.3) with respect to v, we obtain the equation for φ, which for $p \to 0$ is transformed into (5.5), since $\varphi(0, t) \to n(t)$
$$\frac{1}{K_{l_0}}\frac{d\varphi}{dt} = \frac{1}{l!}\varphi^l - \frac{1}{(l-1)!}\varphi n^{l-1}$$

from this, passing to variables $\tau = (n_0 - n)/n_0$, $\psi = \varphi(n/n_0)^{-l/(l-1)}$, $n_0 = n(0)$, we obtain
$$\frac{d\psi}{d\tau} = \psi^l \frac{1}{(l-1)n_0^{l-1}}$$

Consequently
$$\frac{n}{n_0} = \left[1 + \frac{(l-1)^2}{l!}K_{l_0}n_0^{l-1}t\right]^{-1/(l-1)}$$
$$\varphi = \left(\frac{n}{n_0}\right)^{l/(l-1)}\varphi_0(p)\left[1 - \left(1 - \frac{n}{n_0}\right)\frac{\varphi_0^{l-1}}{n_0^{l-1}}\right]^{-1/(l-1)}$$

For the initial distribution defined by $f_0 = Av^{\nu-1}e^{-av}$, where $\nu = (l-1)^{-1}$, $a = \nu n_0/H$ and $A = n_0 a^\nu \Gamma^{-1}(\nu)$, after inverse transformation we obtain
$$f = Av^{\nu-1}\left(\frac{n}{n_0}\right)^{1+\nu}\exp\left(-\frac{avn}{n_0}\right)$$

In the particular case of double collisions $(l = 2)$ this solution becomes the same as the one derived earlier [10, 11].

As the second example, let us take the case of $K_l = K_{l1}\Sigma m_i$ and $K_{l1} = \text{const}$ for the same initial distribution as above. From (5.4) and (1.3), for φ and n we obtain the system
$$\frac{dn}{dt} = -\frac{K_{l1}}{(l-2)!}Hn^{l-1}, \quad \frac{1}{K_{l1}}\frac{\partial\varphi}{\partial t} = -\frac{1}{l!}\frac{\partial\varphi^l}{\partial p} - \frac{1}{(l-1)!}\left[(l-1)Hn^{l-2}\varphi - n^{l-1}\frac{\partial\varphi}{\partial p}\right] \tag{5.6}$$

From the first equation we have
$$\frac{n}{n_0} = \left[1 + \frac{K_{l1}Htn_0^{l-2}}{(l-3)!}\right]^{1/(2-l)} \qquad (l > 2)$$

Using variables $\tau = (n_0 - n)/n_0$ and $\Phi = \varphi/n$, from the second equation we have
$$\frac{\partial\Phi}{\partial\tau} + a(\Phi^{l-1} - 1)\frac{\partial\Phi}{\partial p} = 0 \tag{5.7}$$

Taking into account the initial condition
$$\Phi|_{\tau=0} = A\Gamma(\nu)\frac{1}{(p+a)^\nu}$$

we can write solution (5.7) in the implicit form
$$p = a\tau(\Phi^{l-1} - 1) + \left[\frac{A\Gamma(\nu)}{\Phi n_0}\right]^{l-1} - a$$

from which, after solving for Φ and inverse transformation we have
$$f = \frac{n_0(1-\tau)\nu}{v}\left(\frac{c}{a\tau}\right)^{\nu/2}e^{-a(1+\tau)v}I_\nu(2v\sqrt{ac\tau}), \quad c = [A\Gamma(\nu)n_0^{-1}]^{1/\nu}$$
$$\tau = 1 - \left[1 + \frac{K_{l1}Htn_0^{l-2}}{(l-3)!}\right]^{1/(2-l)}$$

The particular case of (5.6) for $l = 2$ was considered earlier in [12].

To illustrate the structure of moment equations we take as the third example the case of $K_l = K_{li}\Sigma m_{j_1} m_{j_2} \ldots m_{j_i}$. Writing (2.2) as

$$\frac{d}{dt} nQ_q = \frac{K_{li}}{l!}\int \ldots \int \Big(\sum_{(j)} m_{j_1} m_{j_2} \ldots m_{j_i}\Big)\Big(\sum_{(r)} P(r|_1^l) \times m_1^{r_1} m_2^{r_2} \ldots m_l^{r_l}\Big) \prod_{j=1}^{l} f(m_j)\, dm_j$$

we readily observe that generally its right-hand part is expressed in terms of the product of different moments. For $q \geqslant 2$

$$\frac{d}{dt} nQ_q = \frac{K_{li}}{l!} n^l \sum_{(\alpha)} \beta_q Q_1^{\alpha_1} Q_2^{\alpha_2} \ldots Q_q^{\alpha_q}, \quad \beta = \beta(\alpha_1, \ldots, \alpha_q)$$

where summation in the right-hand part is extended over all different sets of integers α_k such that

$$\alpha_1 + \alpha_2 + \ldots + \alpha_q \leqslant l, \quad 1 \cdot \alpha_1 + 2 \cdot \alpha_2 + \ldots + q\alpha_q = i + q$$

The determination of coefficients β is a very cumbersome combinatorial problem. For example, in the particular case of $i = l$ we have

$$\frac{dnQ_2}{dt} = \frac{K_{ll} n^l}{(l-2)!} Q_1^{l-2} Q_2^2$$

$$\frac{dnQ_3}{dt} = \frac{3K_{ll} n^l}{(l-2)!} [Q_1^{l-2} Q_2 Q_3 + (l-2)\, Q_1^{l-3} Q_2^3]$$

$$\frac{dnQ_4}{dt} = \frac{K_{ll} n^l}{(l-2)!} [4Q_1^{l-2} Q_2 Q_4 + 3Q_1^{l-2} Q_3^2 + 18(l-2)\, Q_1^{l-3} Q_2^2 Q_3]$$

6. Self-similar solutions. Let in the absence of diffusion all interactions be defined by homogeneous functions, in which case $q = 0$ and

$$K_s(\alpha m|_1^s) = \alpha^{\mu_s} K_s(m|_1^s), \; F_s(\alpha m|_1^s) = \alpha^{\nu_s} F_s(m|_1^s) \tag{6.1}$$
$$E_{sr}(\alpha m|_1^s; \alpha p|_1^r) = \alpha^{\eta_{sr}} E_{sr}(m|_1^s; p|_1^r)$$

Then in the case of space homogeneity the input kinetic equation (1.3) has self-similar solutions of the form

$$f(v, t) = g(t)\, \psi(V), \qquad V = v / h(t) \tag{6.2}$$

with

$$f(v, 0) = g(0)\, \psi\left(\frac{v}{h(0)}\right)$$

which implies that the form of function $\psi(V)$ is predetermined by the input data. It is shown below that a solution of the form (6.2) exists when special conditions are imposed on K_s, F_s and E_{sr} as well as on initial distributions. The substitution of (6.2) into (1.1) yields

$$n = ghC_0, \qquad H = gh^2 C_1 \tag{6.3}$$

where C_0 and C_1 are arbitrary constants. From this

$$g = n^2 \frac{C_1}{HC_0^2}, \qquad h = \frac{H}{n} \frac{C_0}{C_1} \tag{6.4}$$

Using formulas (6.1) and (6.2), from (1.3) we obtain

$$\frac{dg}{dt}\psi(V)-\frac{g}{h}\frac{dn}{dt}V\frac{d\psi}{dV}=\sum_{s=2}^{\infty}\frac{1}{s!}h^{\mu_s+s-1}g^s(\gamma_{ks}^{+}-s\gamma_{ks}^{-})+ \tag{6.5}$$

$$\sum_{s=2}^{\infty}\frac{1}{s!}h^{\nu_s+s-1}g(s\gamma_{fs}^{+}-\gamma_{fs}^{-})+\sum_{s=2}^{\infty}\sum_{r=2}^{\infty}\frac{1}{s!\,r!}h^{\eta_{sr}+s+r-2}g^s(r\gamma_{esr}^{+}-s\gamma_{esr}^{-})$$

$$\frac{dn}{dt}=\sum_{s=2}^{\infty}\frac{1-s}{s!}(h^{\mu_s+s}g^s\gamma_{ks}-h^{\nu_s+s}g\gamma_{fs})+\sum_{s=2}^{\infty}\sum_{r=2}^{\infty}\frac{r-s}{s!\,r!}h^{\eta_{sr}+s+r-1}g^s\gamma_{esr} \tag{6.6}$$

where

$$\gamma_{ks}^{+}(V)=\int\ldots\int K_s\left(\xi\,|_1^{s-1},V-\sum_1^{s-1}\xi_i\right)\psi\left(V-\sum_1^{s-1}\xi_i\right)\prod_{j=1}^{s-1}\psi(\xi_j)\,d\xi_j \tag{6.7}$$

$$\gamma_{ks}^{-}(V)=\int\ldots\int K_s(\xi\,|_1^{s-1},V)\,\psi(V)\prod_{j=1}^{s-1}\psi(\xi_j)\,d\xi_j$$

$$\gamma_{fs}^{+}(V)=\int\ldots\int F_s(\xi\,|_1^{s-1},V)\,\psi\left(V+\sum_1^{s-1}\xi_i\right)\prod_{j=1}^{s-1}d\xi_j$$

$$\gamma_{fs}^{-}(V)=\int\ldots\int F_s\left(\xi\,|_1^{s-1},V-\sum_1^{s-1}\xi_i\right)\psi(V)\prod_{j=1}^{s-1}d\xi_j$$

$$\gamma_{esr}^{+}(V)=\int\ldots\int E_{sr}\left(\xi\,|_1^{s-1},\sum_1^{r-1}\zeta_i+V-\sum_1^{s-1}\xi_i;\zeta\,|_1^{r-1},V\right)\times$$

$$\psi\left(\sum_1^{r-1}\zeta_i+V-\sum_1^{s-1}\xi_i\right)\prod_{j=1}^{s-1}\psi(\xi_j)\,d\xi_j\prod_{k=1}^{r-1}d\zeta_k$$

$$\gamma_{esr}^{-}(V)=\int\ldots\int E_{sr}\left(\xi\,|_1^{s-1},V;\zeta\,|_1^{r-1},\sum_1^{s-1}\xi_i+V-\sum_1^{r-1}\zeta_i\right)\times$$

$$\psi(V)\prod_{j=1}^{s-1}\psi(\xi_j)\,d\xi_j\prod_{k=1}^{r-1}d\zeta_k$$

$$\gamma_{ks}=\int\ldots\int K_s(\xi\,|_1^s)\prod_{j=1}^{s}\psi(\xi_j)\,d\xi_j,\quad \gamma_{fs}=\int\ldots\int F_s(\xi\,|_1^s)\,\psi\left(\sum_1^s\xi_i\right)\prod_{j=1}^{s}d\xi_j,$$

$$\gamma_{esr}=\int\ldots\int E_{sr}(\xi\,|_1^s;\zeta\,|_1^r)\prod_{j=1}^{s}\psi(\xi_j)\,d\xi_j\prod_{k=1}^{r}d\zeta_k$$

Let us assume that the quantities in (6.7) are independent of time and specify that the coefficients in the right-hand part of (6.5) which depend on time t be proportional to n^{β}; then the coefficients in (6.6) will be proportional to $n^{\beta-1}$. The number β proves to be associated with the degrees of homogeneity of functions in (6.1) by relationships

$$\mu_s=s+1-\beta,\qquad \nu_s=3-s-\beta,\qquad \eta_{sr}=s-r+2-\beta$$

which is possible when $\mu_s=\nu_s+2(s-1)=\eta_{sr}+r-1$. Using (6.4) we transform (6.5) and (6.6) to the form

$$\frac{dn}{dt}\frac{C_1}{HC_0^2}\left[2\psi + V\frac{d\psi}{dV}\right] = n^{\beta-1}\left(\frac{HC_0}{C_1}\right)^{-\beta}\left\{\sum_{s=2}^{\infty}\left(\frac{H}{C_1}\right)^s \times \right. \tag{6.8}$$
$$\frac{1}{s!}(\gamma_{ks}^{+} - s\gamma_{ks}^{-}) + \sum_{s=2}^{\infty}\frac{1}{s!}\left(\frac{H}{C_1}\right)(s\gamma_{fs}^{+} - \gamma_{fs}^{-}) +$$
$$\left.\sum_{s=2}^{\infty}\sum_{r=2}^{\infty}\frac{1}{s!\,r!}\left(\frac{H}{C_1}\right)^s (r\gamma_{esr}^{+} - s\gamma_{esr}^{-})\right\}$$

$$\frac{dn}{dt} = n^{\beta-1}\left(\frac{HC_0}{C_1}\right)^{1-\beta}\left\{\sum_{s=2}^{\infty}\frac{1-s}{s!}\left[\left(\frac{H}{C_1}\right)^s \gamma_{ks} - \right.\right. \tag{6.9}$$
$$\left.\left.\frac{H}{C_1}\gamma_{fs}\right] + \sum_{s=2}^{\infty}\sum_{r=2}^{\infty}\left(\frac{H}{C_1}\right)^s \frac{r-s}{s!\,r!}\gamma_{esr}\right\}$$

Eliminating from these formulas n we obviously obtain an integro-differential equation for $\psi(V)$, whose solution yields the admissible initial distributions. The self-similar solution itself is of the form (6.2), where g and h are expressed in terms of n in conformity with (6.4), and n satisfies Eq. (6.9). A solution coinciding with the one considered here for the particular case of (1.9) was previously obtained in [13, 14].

Note that self-similar solutions exist also in the presence of diffusion with a special form of stream $\mathbf{q}$. The exact solutions derived in Sects. 5 and 6 can be extended by transformation of the time variable to the case, when functions K_s, F_s and E_{sr} depend not only on m_i amd p_i but contain a coefficient (one and the same) which explicitly contains time.

The case of space inhomogeneity requires special consideration; however in the case, important from the practical point of view, in which

$$\mathbf{u} = u(t, y)\,\mathbf{e}_x, \quad f = f(v, t, y), \quad K_s = K_s(m|_1^s;\ t, y)$$
$$F_s = F_s(m|_1^s;\ t, y), \quad E_{sr} = E_{sr}(m|_1^s;\ p|_1^r;\ t, y)$$

the reasoning in Sects. 5 and 6 remains valid, and the coordinate y appears in the solution as a parameter.

7. Concluding remarks. A more detailed definition of suspension would be obtained by the introduction of the distribution function $f^*(v, t, \mathbf{r}, \xi)$, where ξ is the velocity of the aggregate. Then

$$f = \int f^* d\xi, \quad \mathbf{q} = \int (\bar{\mathbf{u}} - \xi) f^* d\xi$$

Functions $\Gamma_k^{\pm}$, $\Gamma_f^{\pm}$ and $\Gamma_e^{\pm}$ are generally related to the integral of collisions and equations for f^*. The form of that equation is not known and the available data on the interaction of particles of a concentrated suspension between themselves and with the carrier fluid are insufficient for its derivation. A kinetic equation in its conventional meaning apparently does not exist for f^*, since the order of magnitude of the time of dynamic interaction of aggregates is the same as that of function f^* relaxation. If, however, the time of aggregate interaction is considerably shorter than the relaxation time of f, it is possible to construct for it directly equations of the kind (1.3).

A similar analysis can be carried out for suspensions containing aggregates of several different kinds. This is done by introducing a system of distribution functions $f_\alpha(v, t, \mathbf{r})$, $\alpha = 1, 2, \ldots$, and by including in the right-hand parts of kinetic equations of terms correponding to interactions with transformation of one kind of suspension into another.

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